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1989 J. Phys. A: Math. Gen. 22 L179

(http://iopscience.iop.org/0305-4470/22/5/008)

J. Phys. A: Math. Gen. **22** (1989) L179–L183. Printed in the UK

LETTER TO THE EDITOR

Holonomy and lattice simulation of topologically non-trivial gauge fields

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Received 23 November 1988

Abstract. The holonomy of a gauge configuration is calculated from the transition function and the path-ordered exponential of the gauge potential for the case of general base manifolds. A general rule for the lattice simulation of gauge fields is proposed by explicitly giving the link variables. It is applied to a cold-start Monte Carlo study of the U(1) gauge field on a torus, and the average energy of a definite topological configuration is calculated.

In lattice gauge theories, each link variable $U(b)$ independently takes values in the structure group G (Wilson 1974). The ordered product

$$U(b_k)U(b_{k-1})\ldots U(b_1) \tag{1}$$

then gives the holonomy along a closed loop B which is composed of links $b_1, b_2, \ldots, b_k$. A typical lattice action is expressed by a sum of small holonomy elements along plaquettes:

$$S=\sum_p \frac{\tau}{\Delta}\left(1-\frac{1}{N}\,\mathrm{Re\ Tr}\ UU\ldots U\right) \tag{2}$$

where the sum is taken over all plaquettes, Δ is the area of plaquette p, τ is the volume of the dual cell to p, and N is the dimension of the matrix representation of U. Various thermal quantities of the system at temperature T are computed from the probability weight of $\exp(-S/T)$. Now, if the base manifold has the topology of R^n, it is covered by only one coordinate patch. Then the link variables $U(b)$ are trivially simulated by the gauge potential A. If the base space is an n-torus, n-sphere or any other space with non-trivial topology, however, the lattice simulation is no longer a trivial task. It is in precisely this case that many topologically distinct gauge configurations arise. If one obtains statistical equilibrium through a randomly started Monte Carlo iteration, the result obtained is a mixture of various topological excitations, each of which corresponds to a metastable state with a definite topological charge. It is important to choose one of them and calculate its thermal properties by a cold-start Monte Carlo iteration.

In this letter we consider how we can give the link variables $U(b)$ on a latticised manifold in the case that the base manifold is not covered by a single coordinate patch. These link variables must, as a whole, simulate the prescribed gauge configuration. The guiding principle is that the ordered product (1) should give the correct holonomy. As an immediate application, a cold-start Monte Carlo calculation for the U(1) gauge field on a two-dimensional flat torus will be made.

0305-4470/89/050179+05$02.50

As a preliminary step let us introduce the notion of a path-ordered exponential of a Lie-algebra-valued 1-form along a curve. Let G and L(G) be a Lie group and its Lie algebra. Let Y be a mapping from the unit interval $I=[0,1]$ to L(G). Let a be a curve in G determined by equations

$$R(a(t)^{-1})a'(t) \equiv a'(t)a(t)^{-1} = Y(t) \tag{3}$$

$$a(0) = e \tag{4}$$

where e is the identity element of G, $a'(t)$ is a vector tangential to a at $a(t)$ and R denotes the right movement of G. In equation (3) the left-hand side is an element of $T_e(G)$, which is identified with L(G) in the standard way. Hence $a(1)$ is uniquely determined (Kobayashi and Nomizu 1963) and we shall write it as

$$a(1) = P\exp\left(\int_I Y(t)\,\mathrm{d}t\right) \tag{5}$$

and refer to it as the path-ordered exponential of the 1-form $Y(t)\,\mathrm{d}t$ along the interval I.

Let us further introduce a differential manifold M and a differential curve B on M. Let Ω be a Lie-algebra-valued 1-form on M. Then $B^*\Omega$, being the pullback of Ω on the unit interval, is a Lie-algebra-valued 1-form on I. Consider the equations

$$a'(t)a(t)^{-1} = \Omega(B'(t)) \tag{6}$$

$$a(0) = e. \tag{7}$$

The unique solution for $a(1)$ is

$$\begin{aligned} a(1) &= P\exp\left(\int_I \Omega(B'(t))\,\mathrm{d}t\right) \\ &= P\exp\left(\int_I B^*\Omega\right). \end{aligned} \tag{8}$$

We shall write it in another form:

$$a(1) = P\exp\left(\int_B \Omega\right). \tag{9}$$

It does not depend on the parametrisation of B.

Turning to the calculation of the holonomy, we shall first work with a single patch. Suppose that a connection 1-form ω is defined on a principal G bundle $P(M^n, G)$ over the n-dimensional base manifold M^n. Let V_1 be a coordinate neighbourhood of M^n and let φ_1 be its coordinate function so that

$$\begin{aligned} &\varphi_1 : \pi^{-1}(V_1) \to V_1 \otimes G \\ &\varphi_1(u) = (\pi u, \varphi_{1,x}(u)) \qquad x = \pi u \end{aligned} \tag{10}$$

where π is the canonical projection. A local cross section σ_1 is naturally defined with respect to φ_1 as

$$\sigma_1 : V_1 \to \pi^{-1}(V_1) \qquad \sigma_1(x) = \varphi_{1,x}^{-1}(e). \tag{11}$$

The gauge potential A_1 is the pullback of ω by σ_1:

$$A_1 = \sigma_1^*\omega \tag{12}$$

which is a Lie-algebra-valued 1-form on V_1. Consider a smooth curve B_1 in V_1 such that $B_1(0)=x_0$ and $B_1(1)=x_1$. Let $v_1=\sigma_1\circ B_1$, which is a curve on the cross section σ_1. There is a unique horizontal lift C_1 of B_1 which starts at $v_1(0)=\varphi_{1,x_0}^{-1}(e)$. This is given in the form

$$C_1(t)=v_1(t)a(t)\equiv R(a(t))v_1(t) \tag{13}$$

where a is a curve in G to be determined. From the horizontality of C_1 we have $\omega(C_1'(t))=0$; $a(t)$ satisfies the differential equation (Kobayashi and Nomizu 1963)

$$\begin{aligned} a'(t)a(t)^{-1} &= -\omega(v_1'(t)) \\ &= -A_1(B_1'(t)) \end{aligned} \tag{14}$$

which is uniquely solved and gives

$$a(1)=P\exp\left(\int_{B_1}-A_1\right). \tag{15}$$

Then the parallel transport of $C_1(0)$ along B_1 is

$$C_1(1)=\varphi_{1,x_1}^{-1}(e)P\exp\left(\int_{B_1}-A_1\right). \tag{16}$$

Because parallel transport commutes with the right movement of fibre, the parallel transport of $u_0\in\pi^{-1}(x_0)$ along B_1 is given by

$$u_1=\varphi_{1,x_1}^{-1}(e)P\exp\left(\int_{B_1}-A_1\right)\varphi_{1,x_0}(u_0). \tag{17}$$

Mapping both sides by φ_{1,x_1}, we have

$$\varphi_{1,x_1}(u_1)=P\exp\left(\int_{B_1}-A_1\right)\varphi_{1,x_0}(u_0). \tag{18}$$

Next we shall take into account other patches. Let B be a closed curve in M^n starting from and ending at x_0. Suppose that k coordinate neighbourhoods $V_1, V_2, \ldots, V_k$ are necessary to cover the loop B. Let us divide B into k pieces so that $B=B_k\circ B_{k-1}\circ\ldots\circ B_1$, and arrange things so that

$$B_i\subset V_i \qquad B_i(0)=x_{i-1} \qquad B_i(1)=x_i \qquad x_{i-1}\in V_{i-1}\cap V_i \tag{19}$$

for $i=1,2,\ldots,k$ with $x_0=x_k$ and $V_0=V_k$. The definitions of φ_i, $\varphi_{i,x}$, σ_i and A_i are analogous to the $i=1$ case. Let the parallel transport of $u_{i-1}\in\pi^{-1}(x_{i-1})$ along B_i be u_i for $i=1,2,\ldots,k$. Then

$$\varphi_{i,x_i}(u_i)=P\exp\left(\int_{B_i}-A_i\right)\varphi_{i,x_{i-1}}(u_{i-1}). \tag{20}$$

Successively applying (20) we see that the parallel transport of u_0 along B is given by

$$\begin{aligned} \varphi_{k,x_k}(u_k) = {} & P\exp\left(\int_{B_k}-A_k\right)g_{k,k-1}(x_{k-1})P \\ & \times\exp\left(\int_{B_{k-1}}-A_{k-1}\right)\ldots g_{2,1}(x_1)P\exp\left(\int_{B_1}-A_1\right)\varphi_{1,x_0}(u_0) \end{aligned} \tag{21}$$

where we used the definition of the transition function

$$g_{i,j}(x)\coloneqq\varphi_{i,x}(u)\varphi_{j,x}(u)^{-1} \qquad x=\pi u\in V_i\cap V_j. \tag{22}$$

Mapping both sides by $g_{1,k}(x_0)$ we obtain

$$\varphi_{1,x_0}(u_k) = H\varphi_{1,x_0}(u_0) \tag{23}$$

$$H = g_{1,k}(x_0)P\exp\left(\int_{B_k} -A_k\right)g_{k,k-1}(x_{k-1})\ldots g_{2,1}(x_1)P\exp\left(\int_{B_1} -A_1\right). \tag{24}$$

The group element H precisely gives the holonomy along B.

The above form for the holonomy suggests the rule for $U(b)$ as follows. Suppose that M^n is covered by the patches $V_1, V_2, \ldots$. We can arrange them so that the following three conditions are met.

(i) Each site (lattice point) is contained in one and only one patch.

(ii) Let b be a link. If $b(1) \in V_i$ and $b(0) \in V_i$, then the link b itself is contained in V_i. Put $b' = b(\frac{1}{2})$. We shall call this case A.

(iii) If $b(0) \in V_i$, $b(1) \in V_j$ and $i \neq j$, we can choose a point b' on b such that $b' \in V_i \cap V_j$. Then $g_{j,i}(b')$ is well defined. We shall call this case B.

Now our rule is: for case A, put

$$U(b) = \exp\left(-\int_b A_i\right) \doteqdot \exp(-A_i(b')_\mu b^\mu) \tag{25}$$

and for case B, put

$$U(b) = g_{j,i}(b'). \tag{26}$$

The $U(b)$ of case A are very close to the identity element, because b is a small vector. So we can call them 'small phases'. On the other hand, the $U(b)$ of case B cannot be chosen arbitrary close to the identity. Hence they are 'big phases'.

As an immediate application of the above rule, let us consider the statistical mechanics of the U(1) gauge model on a two-dimensional flat torus. A point on a torus is parametrised by a pair of real numbers (x, y), where $0 \leqslant x < 1$ and $0 \leqslant y < 1$. Figure 1 shows that a hot-started Monte Carlo iteration gives various configurations, each of which has a distinct topological charge defined by

$$\Phi = \sum_p \operatorname{Im} UU\ldots U. \tag{27}$$

The base space is covered by at least four coordinate patches. Fortunately, however, by successively applying gauge transformations, we arrive at only one gauge potential, which is smoothly defined everywhere except a circle $C = \{(x, 0) \mid 0 \leqslant x < 1\}$, on which

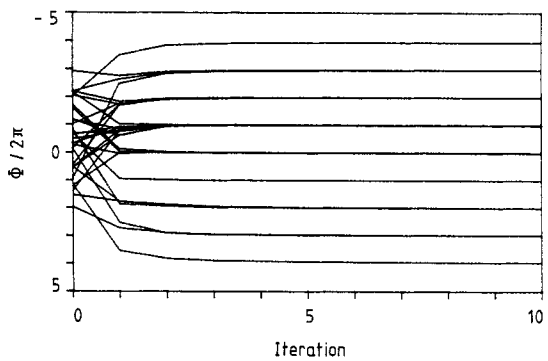

Figure 1. Plot of $\Phi/2\pi$ for $T = 0.1$ in a random-start Monte Carlo iteration.

the transition function is defined. Thus a U(1) gauge potential is continuously deformable to the following potential:

$$A_x = -\mathrm{i}By \qquad B = \text{constant}$$
$$A_y = 0 \tag{28}$$

which has a discontinuity on the circle C where the transition function is defined as

$$g = \exp(-\mathrm{i}Bx). \tag{29}$$

From the univalence of g, we have $B = 2\pi m$ with an integer m. The vector potential (28) and the transition function (29) can be used for the initial configuration of a cold-start Monte Carlo calculation. Average energy per plaquette is plotted in figure 2.

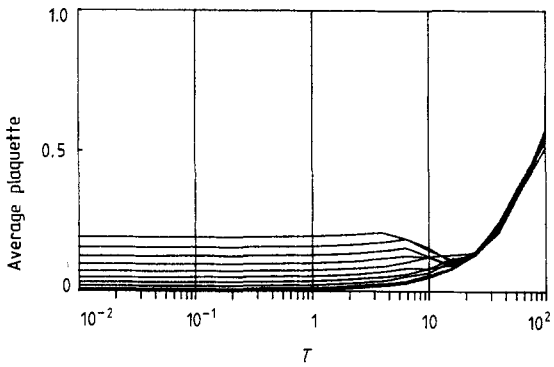

Figure 2. Plot of the average plaquette in a cold-start Monte Carlo calculation. Topological number m ranges from 0 (bottom) to 10 (top).

We see that at sufficiently low temperatures, there are many metastable states and at high temperatures they lose their topology. This topology collapse is due to finite lattice spacing. If we had first taken the limit $\Delta \to 0$, these topological excitations could well survive to arbitrarily high temperatures.

References

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